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Overall moduli and constitutive relations of bodies containing multiple bridged microcracks ☆

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Abstract

The overall moduli and non-linear constitutive relations of solids containing randomly oriented and parallel multiple bridged microcracks are studied in this paper. For both linear and non-linear bridging laws, two non-dimensional mixed parameters which play an important role in determining the overall moduli and constitutive properties are identified. It is shown that the overall moduli of a body containing bridged cracks can be derived from those of the body containing unbridged cracks of the same configuration with the crack density parameter being discounted by these two non-dimensional parameters. For non-linear bridging, attention is paid to the power-law bridging which represents the existing bridging laws in the literature. Particular emphasis is placed on the widely used square-root type bridging law. In this case, closed-form expressions for the overall non-linear constitutive relations are obtained. For a general power-law bridging, it is found that the overall constitutive behaviour of the cracked body is governed by Ramberg–Osgood type relations under strong bridging. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Overall moduli; Constitutive relations; Bridged cracks; Pseudo-traction

1. Introduction

Bridging is an important mechanism in the deformation of solids. In fibre-reinforced composite materials made of brittle or quasi-brittle matrices and subjected to tension, multiple cracking is usually the first sign of damage. The appearance of multiple bridged cracks causes the stress–strain curve to deviate from linearity and thus gives the material a strain-hardening character. In these materials, bridging of multiple microcracks by fibres plays an essential role in increasing their toughness and in preventing a sudden loss of their load carrying capacity when the microcracks coalesce and localise into a large band. In polymers, the craze zones can also be regarded as multiple bridged cracks (e.g. Kausch, 1987). At the atomic level, the atomic bond between two lattice planes in a solid can also be represented by a non-linear bridging law that depicts the relation between the attractive stress and the separation (Bao and Suo, 1992).

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On the macroscopic level, a lot of models have been developed to predict the overall properties of solids containing unbridged cracks and other forms of flaws (Kachanov, 1992; Nemat-Nasser and Hori, 1999). On the micromechanical level, the reinforcing and toughening effect of fibres on cracks in fibre-reinforced advanced composite materials have also been studied intensively in the context of fracture mechanics (e.g. Marshall et al., 1985; Rose, 1987; Nemat-Nasser and Hori, 1987; Chiang et al., 1993; Cox and Rose, 1996, etc.). However, there have been relatively few studies on the macroscopic constitutive behaviour of fibre-reinforced composites, such as cementitious composites, and solids containing multiple bridged microcracks. Wang et al. (1986a, 1986b) studied in detail the damage process in short fibre-reinforced composite materials. They measured the crack configuration parameters and predicted the effective modulus of the materials containing multiple microcracks using the self-consistent approach. However, in their study, the bridging effect of the fibres on the cracks is ignored. As pointed by Nemat-Nasser (1987), for fibre- or whisker-reinforced composite materials, the bridging effect over the opening cracks must be taken into account in the prediction of the overall properties of the materials. The relation between the macroscopic constitutive response and the micromechanical parameters is also vital in the material design. For quasi-brittle monolithic or fibre-reinforced composites, in which the deformation process leading to complete rupture usually involves multiple cracking, the complete constitutive behaviour is important to the analysis of propagation of a macroscopic crack, as was demonstrated by Ortiz (1988).

Recently, a few studies have been devoted to the prediction of overall properties of solids containing multiple cracks that are subjected to bridging. Vagaggini et al. (1995) have previously studied the constitutive relations of fibre-reinforced ceramic composite materials with evolving cracks. Their constitutive relations are based upon the analysis of deformations caused by a through matrix crack and debonding/sliding between the fibre and the matrix. Thus it is akin to the ACK model (Aveston et al., 1971). Karihaloo et al. (1996) have studied the complete constitutive behaviour of short fibre-reinforced cementitious composites under unidirectional tension. Their study is in a numerical form and is limited to the doubly periodic rectangular array of cracks that are subjected to a linear bridging law. The counterpart closed-form expressions of moduli were recently obtained by Wang et al. (2000a, 2000b) using an approximate asymptotic analysis. In the present paper, first, the overall moduli of bodies containing multiple parallel and randomly distributed microcracks are presented when the cracks are subjected to linear bridging. Two non-dimensional mixed parameters are identified for in-plane shear modulus and tensile modulus. By these two parameters, the relationship between the overall moduli of bodies containing multiple bridged microcracks and those for the unbridged case is established in a concise form. Moreover, by adjusting the values of these two non-dimensional parameters, some interesting results can be obtained. Next, when the microcracks are subjected to non-linear bridging laws, which have been well studied and widely used in the literature, the overall non-linear constitutive relations of materials are investigated. For the widely used square-root type non-linear bridging law, the overall non-linear constitutive relations of the cracked body are obtained in a closed-form. For general power-law bridging, the overall constitutive relations are also obtained in closed-form expressions under strong bridging, which are shown to be of the Ramberg–Osgood type. Finally, the non-linear analysis is applied to the doubly periodic arrays of multiple bridged cracks where the interaction among the cracks is taken into account explicitly. The present study serves as a bridge between the micromechanical analyses of bridging mechanisms and the macroscopic behaviour of solids characterised by multiple bridged microcracks.

2. General formulae

The overall strain and stress of a body containing multiple flat cracks are related via

$$\varepsilon_{ij} = D^0_{ijkl}\sigma_{kl} + \frac{1}{2V}\sum_n \int_{S_n} ([u_i]n_j + [u_j]n_i)\,\mathrm{d}S_n \tag{1}$$

where σ_{ij} and ε_{kl} are the average stress and strain components, respectively. $[u_i]$ and n_i are the total crack opening/sliding displacement (COD/CSD) and the component of the unit vector normal to the crack faces. S_n is the region occupied by the crack. D^0_{ijkl} is the compliance tensor of the uncracked material. The relation (1) can also be written as (Nemat-Nasser and Hori, 1999)

$$\varepsilon_{ij} = D^0_{ijkl}\sigma_{kl} + H_{ijkl}\sigma_{kl} \tag{2}$$

where the supplementary tensor $\mathbf{H}$ can be directly calculated from the expression

$$H_{ijkl}\sigma_{kl} = \frac{1}{2V}\sum_n \int_{S_n} ([u_i]n_j + [u_j]n_i)\,\mathrm{d}S_n \tag{3}$$

As will be made clear below, when the microcracks are subjected to a non-linear bridging law, the overall behaviour of the cracked body will also be non-linear. Thus the supplementary fourth-order tensor $\mathbf{H}$ will no longer be a constant tensor. It will depend upon the applied stress σ_{ij} besides its usual dependence upon the crack configuration and density. In this case, the overall secant compliance tensor of the material becomes

$$\bar{\mathbf{D}} = \mathbf{D}^0 + \mathbf{H}(\boldsymbol{\sigma}) \tag{4}$$

For multiple unbridged/bridged cracks, the pseudo-traction technique has been shown to be a useful tool in the calculation of the crack opening/sliding displacement (Horii and Nemat-Nasser, 1985; Kachanov, 1987; Hu et al., 1994; Karihaloo et al., 1996). The pseudo-traction technique usually results in a system of integral equations for the unknown pseudo-tractions on the crack faces. Recently, Wang et al. (2000a, 2000b) solved the integral equations based upon an asymptotic analysis for periodic arrays of bridged cracks. This procedure leads to closed-form expressions of the pseudo-tractions and the overall moduli of the cracked bodies. They show that the approximate solutions give very accurate prediction of the overall moduli of the cracked bodies compared with the numerical solutions for low to moderate levels of crack density. In addition, it is shown in the works of Wang et al. (2000a, 2000b) that the overall moduli of the body containing bridged cracks are less sensitive to both the crack configuration and the density than those for the unbridged case.

For multiple bridged cracks, the stress-consistency condition on each crack is written as

$$\sigma^{\mathrm{p}}_{ij}(\mathbf{x}_n) - \sum_{m\neq n}\int_{S_m} K_{ijkl}(\mathbf{x}_n, \mathbf{x}_m)\sigma^{\mathrm{p}}_{kl}(\mathbf{x}_m)\,\mathrm{d}\mathbf{x}_m + p_{ij}(\mathbf{x}_n) = \sigma^0_{ij} \tag{5}$$

where $\sigma^{\mathrm{p}}_{ij}(\mathbf{x}_n)$ is the pseudo-traction on the faces of a flat crack occupying the region S_n, $p_{ij}(\mathbf{x})$ is the bridging stress, and σ^0_{ij} is the applied stress which is interpreted as the homogeneous stress when the cracks are absent. After obtaining the pseudo-traction, the crack opening/sliding displacement can be calculated through

$$[u_i](\mathbf{x}_n) = \int_{S_n} P_{ik}(\mathbf{x}_n, \mathbf{x}'_n)\sigma^{\mathrm{p}}_{kl}(\mathbf{x}'_n)n_l\,\mathrm{d}S_n \tag{6}$$

The bridging stress is usually expressed as a function of the crack opening/sliding displacement

$$p_{ij}(\mathbf{x}_n) = f_{ij}([u_i](\mathbf{x}_n)) \quad (i \text{ not summed}) \tag{7}$$

There have been some studies of bridging laws in the literature. For example, Marshall et al. (1985), and Bao and Song (1993) presented several non-linear bridging laws for fibre-reinforced solids. In the previous studies of Karihaloo et al. (1996), and Wang et al. (2000a, 2000b), a linear bridging law has been chosen to facilitate the analysis. In the present paper, both linear and non-linear bridging laws will be considered.

3. Linear bridging case

In this section, the overall moduli of bodies containing multiple slit-like cracks which are bridged by a linear law will be investigated. First, we consider the case when the cracks are all parallel to, say, axis 1, in a two-dimensional sense. As usual, it is assumed that the size of the cracks is sufficiently small compared with the size of the body under consideration so that each crack can be regarded as situated in an infinitely extended medium. Within the formalism of pseudo-traction technique, the opening/sliding displacement of each crack is determined by the distributed pseudo-traction on its faces only, while the interaction and bridging effect are taken into account in the solution for the pseudo-traction. Thus, the crack opening/sliding displacement is usually determined through solution of the integral equation (5). This accurate procedure will inevitably need numerical computation and only give the overall properties of the cracked body in a numerical form. In order to be able to get closed-form explicit results, the crack opening/sliding displacement is approximated by what is obtained through the application of the average pseudo-traction over the crack faces. In this case, one obtains

$$[u_i](x_{n1}) = \sqrt{a_n^2 - x_{n1}^2}\,\frac{4}{E'}\bar{\sigma}_{i2}^{\mathrm{p}} \quad (i = 1, 2) \tag{8}$$

where $1/E' = 1/E$ for a plane-stress condition, and $1/E' = (1 - \nu^2)/E$ for a plane-strain condition. $\bar{\sigma}_{i2}^{\mathrm{p}}$ is the average pseudo-traction over the crack faces. a_n is half of the crack length. The accuracy of this approximation will be discussed later. Using a linear bridging law, the bridging stress can be expressed as

$$p_{ij}(\mathbf{x}_n) = k_{ijk}[u_k](\mathbf{x}_n) \quad (ij = 12, 22;\ k = 1, 2) \tag{9}$$

where k_{ijk} is the constant bridging stiffness.

If the interaction among the cracks is neglected, then the stress-consistency condition (5) becomes,

$$\sigma_{ij}^{\mathrm{p}}(\mathbf{x}_n) + k_{ijk}\sqrt{a_n^2 - x_{n1}^2}\,\frac{4}{E'}\bar{\sigma}_{k2}^{\mathrm{p}} = \sigma_{ij}^0 \quad (ij = 12, 22;\ k = 1, 2) \tag{10}$$

It is seen that the above condition cannot be satisfied exactly as the right hand side is a constant. Following the procedure in the works of Wang et al. (2000a, 2000b), the condition (10) is approximated in an average sense as follows

$$\frac{1}{2a_n}\int_{-a_n}^{+a_n}\left[\sigma_{ij}^{\mathrm{p}}(x_{n1}) + k_{ijk}\sqrt{a_n^2 - x_{n1}^2}\,\frac{4}{E'}\bar{\sigma}_{k2}^{\mathrm{p}}\right]\mathrm{d}x_{n1} = \sigma_{ij}^0 \quad (ij = 12, 22;\ k = 1, 2) \tag{11}$$

which becomes, after integration,

$$\bar{\sigma}_{ij}^{\mathrm{p}} + \frac{\pi a_n k_{ijk}}{E'}\bar{\sigma}_{k2}^{\mathrm{p}} = \sigma_{ij}^0 \quad (ij = 12, 22;\ k = 1, 2) \tag{12}$$

where $\bar{\sigma}_{ij}^{\mathrm{p}}$ is the average pseudo-traction. Setting $ij = 12$ and 22, in turn, and assuming that there is no coupling between crack opening (sliding) and shear (normal) bridging traction, Eq. (12) yields

$$\bar{\sigma}_{12}^{\mathrm{p}} = \alpha_n \sigma_{12}^0 \tag{13}$$

$$\bar{\sigma}_{22}^{\mathrm{p}} = \beta_n \sigma_{22}^0 \tag{14}$$

where

$$\alpha_n = \frac{1}{1 + \frac{\pi a_n k_{121}}{E'}}, \qquad \beta_n = \frac{1}{1 + \frac{\pi a_n k_{222}}{E'}} \tag{15}$$

Using the pseudo-tractions in (13) and (14), it is straightforward to obtain the overall in-plane shear modulus and the tensile modulus in direction 2 from Eq. (1), assuming a plane-stress condition,

$$\frac{\bar{\mu}_{12}}{\mu} = \left\{1 + \frac{\pi(1+\kappa)}{4} f'\right\}^{-1} \tag{16}$$

$$\frac{\bar{E}}{E} = \left\{1 + 2\pi f''\right\}^{-1} \tag{17}$$

where

$$f' = \sum_{n=1}^{K} N_n a_n^2 \alpha_n, \qquad f'' = \sum_{n=1}^{K} N_n a_n^2 \beta_n \tag{18}$$

The summation is subjected to the constraint

$$N = \sum_{n=1}^{K} N_n \tag{19}$$

where N is the total number of cracks per unit area. When all the cracks have the same length, say, $2a$, expressions (16) and (17) become

$$\frac{\bar{\mu}_{12}}{\mu} = \left\{1 + \frac{\pi(1+\kappa)}{4} \alpha f\right\}^{-1} \tag{20}$$

$$\frac{\bar{E}}{E} = \{1 + 2\pi\beta f\}^{-1} \tag{21}$$

where $f = Na^2$ is the usual crack density parameter, and

$$\alpha = \frac{1}{1 + \frac{\pi a k_{121}}{E}}, \qquad \beta = \frac{1}{1 + \frac{\pi a k_{222}}{E}} \tag{22}$$

Because α and β are always less than 1, it is seen from Eqs. (20) and (21) that these two non-dimensional mixed parameters have the effect of reducing the crack density compared with the unbridged case. It is obvious that when the bridging stiffness is zero, the above results reduce to those for the unbridged case. For strong bridging, the non-interacting approximation can be expected to produce very accurate results even at high crack densities. In the extreme case when the bridging stiffness tends to infinity, the overall moduli of the cracked body are unaffected by the cracks.

The accuracy of the predicted overall moduli based upon the approximation involved in Eqs. (8) and (11) has been discussed in the work of Wang et al. (2000a) for doubly periodic arrays of unbridged and bridged cracks. It is shown that the results so obtained are very accurate for low to moderate levels of crack density ($f = 0 - 0.25$) for doubly periodic crack arrays even when the neighbouring cracks are very close to each other. This is reasonable as the overall properties of the material are primarily controlled by the average crack opening/sliding displacement. Therefore, the above approximation is expected to be accurate for reasonable values of crack density, especially under strong bridging.

Next, we consider the situation where the cracks are randomly oriented in two dimensions. For this, we shall modify the procedure in the book of Nemat-Nasser and Hori (1999) to suit the bridged case under consideration. First, in the local coordinate system, define a fourth-order tensor $\hat{H}^n_{ijkl}$ through

$$\frac{1}{a_n^2} \int_{-a_n}^{+a_n} \frac{1}{2}([\hat{u}_j]\hat{n}_i + [\hat{u}_i]\hat{n}_j)\, \mathrm{d}\hat{x}_{n1} \equiv \hat{H}^n_{ijkl} \hat{\sigma}^0_{kl} \tag{23}$$

where n denotes the crack number. The quantities with a hat denote those in the local coordinate system for crack n. It is easily shown by using Eqs. (8) and (13)–(14) that

$$\hat{H}^n_{1212} = \hat{H}^n_{1221} = \hat{H}^n_{2112} = \hat{H}^n_{2121} = \alpha_n \frac{\pi}{2E'}, \quad \hat{H}^n_{2222} = \beta_n \frac{2\pi}{E'} \tag{24}$$

All other components are zero. Thus, the supplementary tensor **H** in Eq. (3) for the randomly distributed cracks is obtained through

$$H_{ijkl} = \sum_{n=1}^{K} \left\{ N_n a_n^2 \left[\frac{1}{2\pi} \int_0^{2\pi} Q^n_{ip} Q^n_{jq} Q^n_{kr} Q^n_{ls} \hat{H}^n_{pqrs} \, d\theta_n \right] \right\} \tag{25}$$

where θ_n is the crack orientation angle, and Q^n_{ij} are the components of the usual orthonormal tensor. Since the cracks are randomly oriented, **H** is an isotropic tensor which is expressed as

$$H_{ijkl} = h_1 \delta_{ij}\delta_{kl} + h_2 \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{26}$$

From Eqs. (25) and (26), one obtains

$$4h_1 + 2h_2 = \sum_{n=1}^{K} N_n a_n^2 \hat{H}^n_{2222} = \sum_{n=1}^{K} N_n a_n^2 \beta_n \frac{2\pi}{E'} = f'' \frac{2\pi}{E'} \tag{27}$$

$$2h_1 + 3h_2 = \sum_{n=1}^{K} N_n a_n^2 (\hat{H}^n_{2222} + 2\hat{H}^n_{1212}) = f'' \frac{2\pi}{E'} + f' \frac{\pi}{E'} \tag{28}$$

which yields

$$h_1 = \frac{\pi}{4E'}(f'' - f'), \qquad h_2 = \frac{\pi}{2E'}(f'' + f') \tag{29}$$

Substituting **H** into (4) gives

$$\frac{\bar{\mu}}{\mu} = \left\{ 1 + \frac{\pi(\kappa+1)}{8}(f' + f'') \right\}^{-1} \tag{30}$$

$$\frac{\bar{E}}{E} = \left\{ 1 + \frac{\pi}{4}(f' + 3f'') \right\}^{-1} \tag{31}$$

Generally, $h_1 \neq 0$. This is different from the solution of the unbridged cracks where $h_1 = 0$ (Nemat-Nasser and Hori, 1999). When the bridging stiffnesses are zero, the above results reduce to those in the literature (Benveniste, 1986; Nemat-Nasser and Hori, 1999). Again, when all the cracks have a common length $2a$, the overall properties become

$$\frac{\bar{\mu}}{\mu} = \left\{ 1 + \frac{\pi(\kappa+1)}{8} f(\alpha + \beta) \right\}^{-1} \tag{32}$$

$$\frac{\bar{E}}{E} = \left\{ 1 + \frac{\pi}{4} f(\alpha + 3\beta) \right\}^{-1} \tag{33}$$

When we set α or β to be equal to zero, we obtain the overall properties of materials with certain "internal constraints". If α is set to be zero, then the sliding between the crack faces is prohibited and the cracks can only open. On the other hand, if β is set to be zero, then the opening of the cracks is prohibited and the crack faces can only slide. One immediate example in this class is the compression of a cracked body when friction between the crack faces is neglected. Another example is the deformation of a body that is governed by crack sliding only under either tension or compression. Hence we can get the ratio between the overall compressive modulus and the overall tensile modulus of a body containing bridged frictionless microcracks

$$\frac{\bar{E}_{\text{comp}}}{\bar{E}_{\text{tens}}} = \left\{1 + \frac{\pi}{4} f(\alpha + 3\beta)\right\} \Big/ \left\{1 + \frac{\pi}{4} f\alpha\right\} \tag{34}$$

Setting $\alpha = \beta = 1$ in the above expression gives the ratio between the compressive modulus and tensile modulus of a body containing randomly distributed unbridged frictionless microcracks.

4. Non-linear bridging case

For fibre-reinforced brittle matrix composite materials, Marshall et al. (1985) presented a square-root type non-linear bridging law. Bao and Song (1993) studied in detail the effect of various interface conditions on the bridging law, and presented several non-linear bridging laws. These bridging laws can be well represented by the following power-law

$$p_{ij}(\mathbf{x}_n) = k^0_{ijk}([u_k](\mathbf{x}_n))^{1/m} \quad (ij = 12, 22;\ k = 1, 2;\ k \ \text{ not summed}) \tag{35}$$

where k^0_{ijk} is a material constant depending upon micromechanical material parameters (e.g. Marshall et al., 1985; Bao and Song, 1993). It is noted that the one-dimensional bridging law in the literature (i.e. only exerting normal bridging traction) has been generalised to two dimensions in the formula (35). When $m > 1$, it represents a concave bridging law, whereas when $m < 1$, it represents a convex one. When $m = 2$, it becomes the square-root type non-linear bridging law of Marshall et al. (1985).

As the matrix is still regarded as being linear elastic, the crack opening/sliding displacement is still calculated using the formula (8). This is an advantage of the pseudo-traction technique. Here we only consider the randomly distributed parallel microcracks. Substituting (8) into (35) gives

$$p_{ij}(\mathbf{x}_n) = k^0_{ijk}\left(\frac{4}{E'}\right)^{1/m} (\bar{\sigma}^{\mathrm{p}}_{k2})^{1/m} (a_n^2 - x_{n1}^2)^{1/2m} \quad (ij = 12, 22;\ k = 1, 2;\ k \text{ not summed}) \tag{36}$$

Substituting (36) into the stress-consistency condition in (5) and using the averaging process in (11), we get

$$\bar{\sigma}^{\mathrm{p}}_{ij} + \Lambda_{ijk} D \frac{E}{E^{1/m}} (\bar{\sigma}^{\mathrm{p}}_{k2})^{1/m} = \sigma^0_{ij} \quad (ij = 12, 22;\ k = 1, 2;\ k \text{ not summed}) \tag{37}$$

where Λ_{ijk} and D are two non-dimensional parameters defined as follows

$$\Lambda_{ijk} = \frac{4^{1/m} k^0_{ijk} a_n^{1/m}}{E} \tag{38}$$

$$D = \frac{1}{2a_n^{1+1/m}} \int_{-a_n}^{+a_n} (a_n^2 - x_{n1}^2)^{1/2m}\, \mathrm{d}x_{n1} = \frac{\sqrt{\pi}\,\Gamma\left(1 + \frac{1}{2m}\right)}{2\Gamma\left[\frac{1}{2}\left(3 + \frac{1}{m}\right)\right]}, \quad \frac{1}{m} > -2 \tag{39}$$

In Eq. (39), $\Gamma(x)$ is the gamma function.

Setting $ij = 12$ and 22 in turn, we get

$$\frac{\bar{\sigma}^{\mathrm{p}}_{12}}{\sigma^0_{12}} + A_{nm}\left(\frac{\bar{\sigma}^{\mathrm{p}}_{12}}{\sigma^0_{12}}\right)^{1/m} = 1 \tag{40}$$

$$\frac{\bar{\sigma}_{22}^{\mathrm{p}}}{\sigma_{22}^{0}} + B_{nm}\left(\frac{\bar{\sigma}_{22}^{\mathrm{p}}}{\sigma_{22}^{0}}\right)^{1/m} = 1 \tag{41}$$

where

$$A_{nm} = D\frac{4^{1/m}k_{121}^{0}a_{n}^{1/m}}{E}\left(\frac{E}{\sigma_{12}^{0}}\right)^{(m-1)/m}, \qquad B_{nm} = D\frac{4^{1/m}k_{222}^{0}a_{n}^{1/m}}{E}\left(\frac{E}{\sigma_{22}^{0}}\right)^{(m-1)/m} \tag{42}$$

It is not possible to get general solutions of the above equations for any value of m. However, they can be easily solved when $m = 2$, i.e. for the square-root type bridging law. In this case, the pseduo-tractions are

$$\bar{\sigma}_{12}^{\mathrm{p}} = \sigma_{12}^{0}A_{n2}^{2}\left[\frac{-1+\sqrt{1+4/A_{n2}^{2}}}{2}\right]^{2} \tag{43}$$

$$\bar{\sigma}_{22}^{\mathrm{p}} = \sigma_{22}^{0}B_{n2}^{2}\left[\frac{-1+\sqrt{1+4/B_{n2}^{2}}}{2}\right]^{2} \tag{44}$$

Thus, α_n and β_n in Eqs. (13) and (14) become

$$\alpha_n = A_{n2}^{2}\left[\frac{-1+\sqrt{1+4/A_{n2}^{2}}}{2}\right]^{2} \tag{45}$$

$$\beta_n = B_{n2}^{2}\left[\frac{-1+\sqrt{1+4/B_{n2}^{2}}}{2}\right]^{2} \tag{46}$$

The overall secant moduli of the cracked body are also calculated from expressions (16) and (17) for cracks with different lengths, and (20) and (21) for cracks with the same length. The non-linear stress–strain relations for the latter case are

$$\varepsilon_{12} = \frac{\sigma_{12}^{0}}{2\mu}\left\{1 + \frac{\pi(\kappa+1)}{4}fA_{2}^{2}\left[\frac{-1+\sqrt{1+4/A_{2}^{2}}}{2}\right]^{2}\right\} \tag{47}$$

$$\varepsilon_{22} = \frac{\sigma_{22}^{0}}{E}\left\{1 + 2\pi fB_{2}^{2}\left[\frac{-1+\sqrt{1+4/B_{2}^{2}}}{2}\right]^{2}\right\} \tag{48}$$

where A_2 and B_2 are obtained by setting $a_n = a$ and $m = 2$ in (42). Therefore, these expressions along with (42) give the closed-form non-linear overall stress–strain relations of a material containing microcracks that are subjected to square-root type bridging law.

Generally, the strength of a material is orders of magnitude smaller than its Young modulus. Therefore, when the non-dimensional bridging stiffnesses Λ_{121} and Λ_{222} defined in Eq. (38) are not too small (i.e. strong bridging), we can expect that A_{nm} and B_{nm} in (42) are very large as $m > 1$ (concave bridging). Then we have the approximate solutions of Eqs. (40) and (41) for arbitrary $m > 1$

$$\alpha_n = \frac{1}{A_{nm}^{m}}, \quad \beta_n = \frac{1}{B_{nm}^{m}} \tag{49}$$

It is easy to get the stress–strain relations for such a material under in-plane shear and unidirectional tension, respectively,

$$\varepsilon_{12} = \frac{\sigma_{12}^0}{2\mu}\left[1 + \frac{\pi(\kappa+1)}{4}\sum_{n=1}^{K} N_n a_n^2 \frac{1}{A_{nm}^m}\right] \tag{50}$$

$$\varepsilon_{22} = \frac{\sigma_{22}^0}{E}\left[1 + 2\pi\sum_{n=1}^{K} N_n a_n^2 \frac{1}{B_{nm}^m}\right] \tag{51}$$

For cracks of equal length $2a$, the above expressions reduce to

$$\varepsilon_{12} = \frac{\sigma_{12}^0}{2\mu}\left[1 + \frac{\pi(\kappa+1)}{4} f \frac{1}{A_m^m}\right] \tag{52}$$

$$\varepsilon_{22} = \frac{\sigma_{22}^0}{E}\left[1 + 2\pi f \frac{1}{B_m^m}\right] \tag{53}$$

where A_m and B_m are obtained by setting $a_n = a$ in Eq. (42). By substitution, the above expressions can be rewritten as

$$\varepsilon_{12} = \frac{\sigma_{12}^0}{2\mu} + \pi f\left[D\frac{4^{1/m}k_{121}^0 a^{1/m}(\kappa+1)}{8\mu}\right]^{-m}\frac{(\kappa+1)^m}{(8\mu)^m}\left(\sigma_{12}^0\right)^m \tag{54}$$

$$\varepsilon_{22} = \frac{\sigma_{22}^0}{E} + 2\pi f\left(D\frac{4^{1/m}k_{222}^0 a^{1/m}}{E}\right)^{-m}\left(\frac{\sigma_{22}^0}{E}\right)^m \tag{55}$$

Therefore, when the bridging is governed by a power-law, the overall stress–strain relations are approximately governed by Ramberg–Osgood type constitutive laws in the case of strong bridging.

5. Non-linearly bridged periodic arrays of cracks

Overall properties of media containing periodic microstructures have been paid intensive attention in recently years. Although perfect periodic microstructures may not exist in real materials, the estimate of the overall properties of media containing assumed periodic microstructures has broad application and provides limiting values for actual cases, as pointed out by Nemat-Nasser and Hori (1999). A periodic configuration also provides a unit cell for homogenisation. For crack problems in two dimensions, the unbridged crack arrays have been studied by some researchers (e.g. Delameter et al., 1975; Karihaloo, 1978; Deng and Nemat-Nasser, 1992; Nemat-Nasser et al., 1993). A paradox which exists in some of the previous works was resolved by Karihaloo et al. (1996). For doubly periodic configurations, the interaction among the multiple cracks can be taken into account explicitly and accurately using the pseudo-traction technique. However, the resulting integral equations for the unknown pseudo-tractions have to be solved numerically (Karihaloo et al., 1996). Recently, under the assumption that the cracks are so distributed that some high-order terms related to the crack configuration and density can be neglected, Wang et al. (2000a, 2000b) obtained closed-form expressions for the overall moduli. Comparison with numerical solutions has shown that the approximate analytical solutions are very accurate for low to moderate levels of crack density. In this section, we shall extend the procedure in the works of Wang et al. (2000a, 2000b) to the cases of non-linear power-law bridging.

For doubly periodic rectangular and diamond-shaped arrays of bridged microcracks, the stress-consistency condition (5) can be expressed as (Wang et al., 2000b)

$$\left\{ \begin{array}{c} \phi^{\mathrm{r}} \\ \phi^{\mathrm{d}} \end{array} \right\} \bar{\sigma}_{12}^{\mathrm{p}} + \frac{1}{2a} \int_{-a}^{+a} p_{12}(x_1)\,\mathrm{d}x_1 = \sigma_{12}^0 \tag{56}$$

$$\left\{ \begin{array}{c} \psi^{\mathrm{r}} \\ \psi^{\mathrm{d}} \end{array} \right\} \bar{\sigma}_{22}^{\mathrm{p}} + \frac{1}{2a} \int_{-a}^{+a} p_{22}(x_1)\,\mathrm{d}x_1 = \sigma_{22}^0 \tag{57}$$

where the superscripts r and d denote rectangular array and diamond-shaped array, respectively, and

$$\phi^{\mathrm{r}} = 1 + 4\sin^2\frac{\pi a}{W}\exp\left(-2\frac{H}{W}\pi\right)\left[1 - 2\frac{H}{W}\pi\right] \tag{58}$$

$$\phi^{\mathrm{d}} = 1 - 4\sin^2\frac{\pi a}{W}\exp\left(-2\frac{H}{W}\pi\right)\left[1 - 2\frac{H}{W}\pi\right] \tag{59}$$

$$\psi^{\mathrm{r}} = 1 + 4\sin^2\frac{\pi a}{W}\exp\left(-2\frac{H}{W}\pi\right)\left[1 + 2\frac{H}{W}\pi\right] \tag{60}$$

$$\psi^{\mathrm{d}} = 1 - 4\sin^2\frac{\pi a}{W}\exp\left(-2\frac{H}{W}\pi\right)\left[1 + 2\frac{H}{W}\pi\right] \tag{61}$$

In the above expressions, a is the half crack length, H and W are respectively the distance between two neighbouring rows of cracks and the distance between the centres of two neighbouring cracks in the direction of the crack line.

Substituting the bridging law (36) into (56) and (57) yields

$$\left\{ \begin{array}{c} \phi^{\mathrm{r}} \\ \phi^{\mathrm{d}} \end{array} \right\} \frac{\bar{\sigma}_{12}^{\mathrm{p}}}{\sigma_{12}^0} + A_m\left(\frac{\bar{\sigma}_{12}^{\mathrm{p}}}{\sigma_{12}^0}\right)^{1/m} = 1 \tag{62}$$

$$\left\{ \begin{array}{c} \psi^{\mathrm{r}} \\ \psi^{\mathrm{d}} \end{array} \right\} \frac{\bar{\sigma}_{22}^{\mathrm{p}}}{\sigma_{22}^0} + B_m\left(\frac{\bar{\sigma}_{22}^{\mathrm{p}}}{\sigma_{22}^0}\right)^{1/m} = 1 \tag{63}$$

For $m = 2$, the above equations can be solved exactly to give

$$\bar{\sigma}_{12}^{\mathrm{p}} = \left\{ \begin{array}{c} \alpha^{\mathrm{r}} \\ \alpha^{\mathrm{d}} \end{array} \right\} \sigma_{12}^0 \tag{64}$$

$$\bar{\sigma}_{22}^{\mathrm{p}} = \left\{ \begin{array}{c} \beta^{\mathrm{r}} \\ \beta^{\mathrm{d}} \end{array} \right\} \sigma_{22}^0 \tag{65}$$

where

$$\alpha^{\mathrm{r}} = \left[\frac{-A_2 + \sqrt{A_2^2 + 4\phi^{\mathrm{r}}}}{2\phi^{\mathrm{r}}}\right]^2, \qquad \alpha^{\mathrm{d}} = \left[\frac{-A_2 + \sqrt{A_2^2 + 4\phi^{\mathrm{d}}}}{2\phi^{\mathrm{d}}}\right]^2 \tag{66}$$

$$\beta^{\mathrm{r}} = \left[\frac{-B_2 + \sqrt{B_2^2 + 4\psi^{\mathrm{r}}}}{2\psi^{\mathrm{r}}}\right]^2, \qquad \beta^{\mathrm{d}} = \left[\frac{-B_2 + \sqrt{B_2^2 + 4\psi^{\mathrm{d}}}}{2\psi^{\mathrm{d}}}\right]^2 \tag{67}$$

Then the constitutive relations under shearing and tensile loadings are, respectively,

$$\begin{Bmatrix} \varepsilon_{12}^{\mathrm{r}} \\ \varepsilon_{12}^{\mathrm{d}} \end{Bmatrix} = \frac{\sigma_{12}^0}{2\mu}\left[1 + \frac{\pi(\kappa+1)}{4} f \begin{Bmatrix} \alpha^{\mathrm{r}} \\ \alpha^{\mathrm{d}} \end{Bmatrix}\right] \tag{68}$$

$$\begin{Bmatrix} \varepsilon_{22}^{\mathrm{r}} \\ \varepsilon_{22}^{\mathrm{d}} \end{Bmatrix} = \frac{\sigma_{22}^0}{E}\left[1 + 2\pi f \begin{Bmatrix} \beta^{\mathrm{r}} \\ \beta^{\mathrm{d}} \end{Bmatrix}\right] \tag{69}$$

where $f = a^2/(WH)$.

For strong bridging so that the second terms on the left hand sides of the equations (62) and (63) dominate, the constitutive relations revert to the Ramberg–Osgood type ones in Eqs. (54) and (55), in which case the difference between the crack configurations vanishes. The overall behaviour is only dependent upon the crack density parameter f and the bridging stiffnesses. This phenomenon has also been observed for linear bridging case in the work of Wang et al. (2000b).

6. Conclusions

In this paper, the overall properties of bodies containing multiple bridged microcracks are studied. For linear bridging, the overall shear and tensile moduli are presented for parallel and randomly oriented cracks using the non-interaction approximation. It is found that the effect of bridging is equivalent to a reduction in the crack density parameter compared with the unbridged case. For power-law bridging, the unknown pseudo-tractions are found to be the solutions of two non-linear equations. The overall constitutive relations of the cracked body are given in closed-form expressions for the square-root type bridging law. For strong bridging, it is found that the overall constitutive behaviour of the cracked body can be approximately described by Ramberg–Osgood type relations under general power-law bridging. For both linear and non-linear bridging, two non-dimensional mixed parameters, namely, α (α_n) and β (β_n), are identified. They are constants for the former case and functions of the applied stress for the latter. The crack opening/sliding stress is reduced by these two parameters compared with unbridged cracks. Consequently, their effect is to reduce the conventional crack density parameter in the expressions of the overall moduli. It is noted that there are various schemes in the literature to take into account the crack interaction for dense microcrack distribution that are not bridged, for example, the self-consistent scheme and differential scheme. In this case, the corresponding moduli or constitutive relations for bridged microcracks may be obtained by replacing the conventional crack density parameter with the reduced crack density parameter.

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